2020

MATHEMATICS — HONOURS

Paper : CC-2

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Choose the correct alternative with proper justification, 1 mark for correct answer and 1 mark for justification : 2×10

(a) The mapping $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n - (-1)^n$, $n \in \mathbb{N}$ is

(i) not a 1-1 mapping (ii) not an onto mapping

(iii) not a mapping (iv) 1-1 and onto mapping.

(b) If $5x + 3 \equiv 5 \pmod{11}$ then one possible value of x is

(i) -7 (ii) 9 (iii) 8 (iv) 7.

(c) A relation R from $\{1,2,...,10\}$ to $\{1,2,...,10\}$ is defined by $mR\,n$ if $m^2 + n^2 = 10$. Then R is

(i) $\{(1,3)\}$ (ii) $\{(3,1)\}$

(iii) $\{(1,3), (3,1)\}$ (iv) $\{(1,3), (-1,3), (1,-3), (-1,-3)\}$.

(d) The range of the function $f(x) = ([n])^2$, $x \in \mathbb{R}$ is

(i) $\mathbb{N}$ (ii) $\mathbb{Z}$

(iii) $\{1, 4, 9, ...\}$ (iv) $\{0, 1, 4, 9, ...\}$.

(e) The value of β such that the rank of the matrix $A = \begin{bmatrix} 0 & \alpha & -\alpha \\ \beta & 0 & 0 \\ 1 & -\alpha & \alpha \end{bmatrix} (\alpha \neq 0)$ is 2, is

(i) 2 (ii) 1 (iii) -2 (iv) -1.

(f) If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$, then the value of $\sum \frac{1}{\alpha^2}$ is

(i) $\frac{2q - r^2}{s}$ (ii) $\frac{2qs - r^2}{s}$

(iii) $\frac{r^2 - 2q}{s}$ (iv) $\frac{r^2 - 2qs}{s}$.

(g) The equation whose roots are double of the roots of the equation $32x^3 - 14x + 3 = 0$ is

(i) $4x^3 - 7x + 3 = 0$ (ii) $64x^3 - 28x + 6 = 0$

(iii) $3x^3 - 7x + 4 = 0$ (iv) $16x^3 - 7x + 6 = 0$.

(h) The unit digit in 7^{99} is

(i) 7 (ii) 9 (iii) 3 (iv) 1

(i) Consider the set $A = \left\{ z \in \mathbb{C} : \bar{z} = \frac{1}{z} \right\}$, then the points of A lies

(i) on a circle (ii) on a hyperbola

(iii) on an ellipse (iv) on a straight line

(j) The system of equations
$$x + y + z = kx$$
$$x + y + z = ky$$
$$x + y + z = kz$$
will have non-trivial solutions if the values of k are

(i) 3, 0 (ii) 3, –3 (iii) 0, –3 (iv) 1, 0

2. Answer ***any four*** questions : 5×4

(a) By Sturm method prove that the roots of the equation $x^3 - (a^2 + b^2 + c^2)x - 2abc = 0$ are all real.

(b) If α, β, γ be the roots of the equation $x^3 + qx + r = 0$, find the equation whose roots are $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$, $(\alpha-\beta)^2$.

(c) Solve the difference equation $u_{x+2} + u_{x+1} - 12u_x = 7x$, $x \geq 1$.

(d) If a, b, c are positive numbers such that $a + b + c = 1$, then show that
$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} < 5.$$

(e) Solve $z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ in the field of complex numbers.

(f) If $\tan(\theta + i\phi) = \sin(\alpha + i\beta)$, prove that $\sin 2\theta \cot\alpha = \sin h 2\phi \cot h\beta$.

(g) (i) Apply Descartes' rule of signs to determine the possible nature of the roots of the equation $x^7 - 3x^3 - x + 1 = 0$.

(ii) Solve the equation by Ferrari's method : $x^4 + 3x^3 + 5x^2 + 4x + 2 = 0$. 1+4

3. Answer ***any four*** questions : 5×4

(a) (i) Let X be a non-empty set. Prove that the following conditions are equivalent :

(A) ρ is an equivalence relation on X.

(B) ρ is a reflexive relation on X and for all $x, y, z \in X$, if $x \,\rho\, y$ and $x \,\rho\, z$, then $y \,\rho\, z$.

(ii) Let R be a relation on a set A. Define $\tau(R) = R \cup R^{-1} \cup \{(x,x) : x \in A\}$, show that $\tau(R)$ is reflexive and symmetric. 3+2

(b) (i) If $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x + 1$ and $g(x) = \max.\{0, x - 1\}$ for $x \in \mathbb{N}$, then show that $gof = I_{\mathbb{N}}$ but $fog \neq I_{\mathbb{N}}$ where $I_{\mathbb{N}}$ is an identity function on $\mathbb{N}$.

(ii) If the function $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ is defined by $f(x) = 2x$ for all $x \in \mathbb{Z}_5$, then find $f^{-1}([3])$, where $\mathbb{Z}_5$ is the set of all equivalence classes on $\mathbb{Z}$ corresponding to the equivalence relation modulo 5. 3+2

(c) Let $P = \{x \in \mathbb{R} : 0 < x < 1\}$ and $f : P \to \mathbb{R}$ be defined by $f(x) = \dfrac{2x-1}{1-|2x-1|}$. Is f bijective? Justify. If so, find f^{-1}. 3+2

(d) (i) If p be prime and k be a (+ve) integer, then prove that $\phi\left(p^k\right) = p^k\left(1 - \dfrac{1}{p}\right)$.

(ii) If a is relatively prime to b, prove that a^2 is also relatively prime to b. 3+2

(e) Let P be the set of all positive divisors of 36. On P define a relation ρ by : for $a, b \in P$, aPb if and only if $a \mid b$. Prove that (P, ρ) is a poset. Is (P, ρ) a linear ordered set? Justify your answer. 3+2

(f) If p is a prime number such that $p \geq 5$, then prove that $p^2 - 1$ is divisible by 24. 5

(g) Using Chinese remainder theorem solve the following system of congruence equations 5

$$2x \equiv 1 \pmod 3$$
$$5x \equiv 4 \pmod 4 \cdot$$

4. Answer ***any one*** question : 5×1

(a) Check the consistency of the system of equations

$$\begin{aligned} 2x - y + z &= 4 \\ 3x - y + z &= 6 \\ 4x - y + 2z &= 7 \\ -x + y - z &= 9\ . \end{aligned}$$

(b) Reduce the following matrix in the row reduced echelon form :

$$\begin{bmatrix} 1 & 3 & 0 & 5 & 2 \\ 0 & 0 & 3 & 4 & 0 \\ 7 & 1 & 0 & 4 & 1 \\ 5 & 3 & 2 & 1 & 6 \end{bmatrix}.$$

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